
The quadratic scalar radius of the pion and the mixed $\pi - K$ radius*

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Abstract

We consider the quadratic scalar radius of the pion, $\langle r_{S,\pi}^2 \rangle$, and the mixed $K - \pi$ scalar radius, $\langle r_{S,K\pi}^2 \rangle$. With respect to the second, we point out that the more recent (post-1974) experimental results in K_{l3} decays imply a value, $\langle r_{S,K\pi}^2 \rangle = 0.31 \pm 0.06 \text{ fm}^2$, which is about 2σ above estimates based on chiral perturbation theory. On the other hand, we show that this value of $\langle r_{S,K\pi}^2 \rangle$ suggests the existence of a low mass $S_{\frac{1}{2}} K\pi$ resonance. With respect to $\langle r_{S,\pi}^2 \rangle$, we contest the central value and accuracy of current evaluations, that give $\langle r_{S,\pi}^2 \rangle = 0.61 \pm 0.04 \text{ fm}^2$. Based on experiment, we find a robust lower bound of $\langle r_{S,\pi}^2 \rangle \geq 0.70 \pm 0.06 \text{ fm}^2$ and a reliable estimate, $\langle r_{S,\pi}^2 \rangle = 0.75 \pm 0.07 \text{ fm}^2$, where the error bars are attainable. This implies, in particular, that the chiral result for $\langle r_{S,\pi}^2 \rangle$ is 1.4σ away from experiment. We also comment on implications about the chiral parameter \bar{l}_4 , very likely substantially larger (and with larger errors) than usually assumed.

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1. Introduction

The quadratic scalar radius of the pion, $\langle r_{S,\pi}^2 \rangle$, and the mixed $K - \pi$ (quadratic) scalar radius, $\langle r_{S,K\pi}^2 \rangle$, are quantities of high interest for chiral perturbation theory calculations, or, more generally, for pion physics. Using chiral perturbation theory to one loop they can be related to meson masses and decay constants:^[1,2]

$$\begin{aligned} \langle r_{S,K\pi}^2 \rangle &= \frac{6}{M_K^2 - M_\pi^2} \left(\frac{f_K}{f_\pi} - 1 \right) + \delta_2; \\ \delta_2 &= - \frac{1}{192\pi^2 f_\pi^2} \left\{ 15h_2(M_\pi^2/M_K^2) + \frac{19M_K^2 + 3M_\eta^2}{M_K^2 + M_\eta^2} h_2(M_\eta^2/M_K^2) - 18 \right\}, \end{aligned} \quad (1.1a)$$

$$h_2(x) = \frac{3}{2} \left(\frac{1+x}{1-x} \right)^2 + \frac{3x(1+x)}{(1-x)^3} \log x;$$

$$\begin{aligned} \langle r_{S,\pi}^2 \rangle &= \frac{6}{M_K^2 - M_\pi^2} \left(\frac{f_K}{f_\pi} - 1 \right) + \delta_3; \\ \delta_3 &= - \frac{1}{64\pi^2 f_\pi^2} \frac{1}{M_K^2 - M_\pi^2} \left\{ 6(2M_K^2 - M_\pi^2) \log \frac{M_K^2}{M_\pi^2} + 9M_\eta^2 \log \frac{M_\eta^2}{M_\pi^2} - 2(M_K^2 - M_\pi^2) \left(10 + \frac{1}{3} \frac{M_\pi^2}{M_\eta^2} \right) \right\}. \end{aligned} \quad (1.1b)$$

The second can also be expressed in terms of the chiral constant \bar{l}_4 ; to one loop,^[1]

$$\langle r_{S,\pi}^2 \rangle = \frac{3}{8\pi^2 f_\pi^2} \left\{ \bar{l}_4 - \frac{13}{12} \right\}. \quad (1.2)$$

Here f_π , f_K and M_π , M_K are the decay constants and masses of pion and kaon; $M_\eta = 547$ MeV is the eta particle mass. We take $M_\pi = 139.57$ MeV (the charged pion mass), but choose an average kaon mass, $M_K = 496$ MeV. From (1.1a,b), Gasser and Leutwyler^[2] obtain the theoretical predictions

$$\langle r_{S,K\pi}^2 \rangle_{\text{GL}} = 0.20 \pm 0.05 \text{ fm}^2, \quad (1.3a)$$

$$\langle r_{S,\pi}^2 \rangle_{\text{GL}} = 0.55 \pm 0.15 \text{ fm}^2; \quad (1.3b)$$

the errors come from estimated higher order corrections.

For $\langle r_{S,K\pi}^2 \rangle$ we have experimental information from the decays K_{l3}^0 and K_{l3}^\pm . For the first the world average of the Particle Data Tables^[3] is

$$\lambda_0 = 0.025 \pm 0.006 \quad [K_{l3}^0] \quad (1.4a)$$

and $\langle r_{S,K\pi}^2 \rangle = 6\lambda_0/M_\pi^2$. For K_{l3}^\pm the four more modern experimental analyses^[4] give the numbers¹

$$\lambda_0 = \begin{cases} 0.062 \pm 0.024 & \text{Artemov et al. (1997)} \\ 0.029 \pm 0.011 & \text{Whitman et al. (1980)} \\ 0.019 \pm 0.010 & \text{Heintze et al. (1977)} \\ 0.008 \pm 0.097 & \text{Braun et al. (1975).} \end{cases} \quad (1.4b)$$

If we average them, which is permissible since they are compatible within errors, we find

$$\lambda_0 = 0.027 \pm 0.007 \quad [K_{l3}^\pm], \quad (1.4c)$$

a value in perfect agreement with (1.4a), to which it should equal if neglecting isospin breaking effects. We compose (1.4a), (1.4c) to get $\lambda_0 = 0.026 \pm 0.005$, and find what we will consider the experimental value for the form factor:

$$\langle r_{S,K\pi}^2 \rangle_{\text{exp.}} = 0.312 \pm 0.070 \text{ fm}^2. \quad (1.5)$$

¹ This is one of the few cases in which the PDT recommend a number difficult to believe. Perhaps influenced by very old determinations (pre-1975) they give the average value 0.006 ± 0.007 , incompatible both with isospin invariance and with the with post-1974 experiments, and which we disregard.

On comparing with a dispersive calculation, (1.5) strongly suggests the existence of a low energy $S_{\frac{1}{2}} K\pi$ resonance. On the other hand, the central value in (1.5) lies clearly outside the error bars of the chiral theory prediction, (1.3a).

There is no direct measurement of $\langle r_{S,\pi}^2 \rangle$. Donoghue, Gasser and Leutwyler^[5] used the two-channel Omnès–Muskhelishvili method and $\pi\pi$ phase shifts to give what is presented as a precise experiment-based estimate; Colangelo, Gasser and Leutwyler review it and, with a minor updating, accept it at present:^[6]

$$\langle r_{S,\pi}^2 \rangle = 0.61 \pm 0.04 \text{ fm}^2. \quad (1.6)$$

It is difficult to believe that the precision and central value in (1.6) hold at the same time. To get these numbers, Donoghue et al. use experimental phase shifts for $\pi\pi$ scattering above the $\bar{K}K$ threshold, where, because one does not measure the process $\bar{K}K \rightarrow \bar{K}K$, the set of measurements is *incomplete* (as proved for example in refs. 7) and where, indeed, different fits give totally different eigenphases (necessary to perform the Omnès–Muskhelishvili analysis), as may be seen explicitly in ref. 8. Moreover, they neglect multipion contributions which, for the electromagnetic form factor of the pion, account for some 6% of the full result.

As a matter of fact, we will give here a new evaluation (which is the main outcome of the present note) and will, in particular, present examples of phases which are compatible with experimental information, as well as with all physical requirements at high energy, and for which the corresponding $\langle r_{S,\pi}^2 \rangle$ is several standard deviations above (1.6). In particular, we find a safe bound, and a reliable estimate:

$$\langle r_{S,\pi}^2 \rangle \geq 0.70 \pm 0.06 \text{ fm}^2, \quad (1.7)$$

$$\langle r_{S,\pi}^2 \rangle = 0.75 \pm 0.07 \text{ fm}^2. \quad (1.8)$$

Moreover, we show that the error bars in (1.8) are *attainable*.

It thus follows that, also for $\langle r_{S,\pi}^2 \rangle$, the errors due to higher orders are underestimated.

2. The Omnès–Muskhelishvili method for form factors and radii

We consider the scalar pion form factor, $F_S(t)$, and the mixed scalar form factor $f_{K\pi}(t)$. We will also discuss the electromagnetic form factor of the pion, $F_\pi(t)$. In terms of these,²

$$\begin{aligned} F_\pi(t) &\underset{t \rightarrow 0}{\simeq} F_\pi(0) \left\{ 1 + \frac{1}{6} \langle r_\pi^2 \rangle t \right\}, \\ F_S(t) &\underset{t \rightarrow 0}{\simeq} F_S(0) \left\{ 1 + \frac{1}{6} \langle r_{S,\pi}^2 \rangle t \right\}, \\ f_{K\pi}(t) &\underset{t \rightarrow 0}{\simeq} f_{K\pi}(0) \left\{ 1 + \frac{1}{6} \langle r_{S,K\pi}^2 \rangle t \right\}, \end{aligned} \quad (2.1)$$

and $\langle r_\pi^2 \rangle$ is the electromagnetic radius of the pion. For F_π , current conservation gives $F_\pi(0) = 1$. The values of $F_S(0)$, $f_{K\pi}(0)$ may be calculated with chiral dynamics,^[2] but we will not concern ourselves with this here.

Let us denote by $F(t)$ to any of the three form factors in (2.1), and let $\delta(t)$ be its phase:

$$\delta(t) = \arg F(t), \quad t \geq s_{\text{th}}; \quad (2.2)$$

s_{th} is the threshold, $4M_\pi^2$ or $(M_\pi + M_K)^2$, as the case may be. (We note that in (2.2) we do *not* understand the principal value of the argument; the phase has to be taken as varying continuously with t). The Fermi–Watson final state interaction theorem implies that, for $t < s_0$ (where s_0 is the energy at which inelastic

² We define the form factors by

$$\begin{aligned} \langle \pi(p) | J_{\text{e.m.}}^\mu(0) | \pi(p') \rangle &= (2\pi)^{-3} (p^\mu - p'^\mu) F_\pi(t), \\ \langle \pi(p) | m_u \bar{u}u(0) + m_d \bar{d}d(0) | \pi(p') \rangle &= (2\pi)^{-3} F_S(t), \\ \langle \pi(p) | (m_s - m_u) \bar{u}s(0) | K(p') \rangle &= (2\pi)^{-3} f_{K\pi}(t); \end{aligned}$$

the meson states are normalized to $\langle p | p' \rangle = 2p_0 \delta(\mathbf{p} - \mathbf{p}')$, and $t = (p - p')^2$.

channels become nonnegligible), $\delta(t)$ equals a corresponding scattering phase.³ To be precise,

$$\begin{aligned}\delta(t) &= \delta_1(t) \quad [\text{P wave } \pi\pi \text{ phase, for } F_\pi]; \quad s_0 \simeq 1.1 \text{ GeV}^2 \\ \delta(t) &= \delta_0^{(0)}(t) \quad [\text{S0 wave (S wave with isospin 0) } \pi\pi \text{ phase, for } F_S]; \quad s_0 = 4M_K^2 \\ \delta(t) &= \delta_0^{(1/2)}(t) \quad [\text{S}_{\frac{1}{2}} \text{ (S wave with isospin } \frac{1}{2}) \text{ } K\pi \text{ phase, for } f_{K\pi}]; \quad s_0 \simeq 1.5^2 \text{ GeV}^2.\end{aligned}\tag{2.3}$$

We will assume that we know the phases δ_1 , $\delta_0^{(0)}$, $\delta_0^{(1/2)}$, and thus $\delta(t)$, for $t \leq s_0$.

At large t , the Brodsky–Farrar counting rules^[9] imply that

$$F(t) \underset{t \rightarrow \infty}{\simeq} \frac{1}{-t \log^\nu(-t)},\tag{2.4}$$

from which it follows that, unless the phase oscillated at infinity, one must have

$$\delta(t) \underset{t \rightarrow +\infty}{\simeq} \pi \left\{ 1 + \nu \frac{1}{\log t/\hat{t}} \right\}.\tag{2.5}$$

In particular, (2.5) implies that $\delta(\infty) = \pi$. For F_π , the Jackson–Farrar calculation^[9] gives

$$F_\pi(t) \underset{t \rightarrow \infty}{\simeq} 12\pi C_F f_\pi^2 \alpha_s(-t)/(-t),$$

hence $\nu = 1$; for the other form factors one cannot prove a similar behavior rigorously in QCD, although it is likely that $\nu = 1$ also here. \hat{t} is a scale; for the electromagnetic form factor, it is $\sim \Lambda^2$, with Λ the QCD parameter, but its precise value is generally not known. Nevertheless, the feature that the limit $\delta(\infty)$ has to be reached *from above*, i.e., that at asymptotic energies $\delta(t)$ is *larger* than π , seems to be general.

We will use the Omnès–Muskhelishvili method,^[10] with only one channel, to solve for F in terms of δ ; it will turn out that the two-channel method is neither necessary nor reliable [the last for the reasons explained after Eq. (1.6)]. According to it, we have that, given the condition (2.5), the phase determines uniquely F : one has

$$F(t) = F(0) \exp \left\{ \frac{t}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\delta(s)}{s(s-t)} \right\}.\tag{2.6}$$

From this we get a simple sum rule for the square radius $\langle r^2 \rangle$ corresponding to $F(t)$:

$$\langle r^2 \rangle = \frac{6}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\delta(s)}{s^2}.\tag{2.7}$$

In general, we will split $\langle r^2 \rangle$ as follows:

$$\langle r^2 \rangle = Q_J(s_0) + Q_\Phi(s_0) + Q_G(s_0).\tag{2.8}$$

Here Q_J is the piece in (2.7) coming from the region where we know δ ,

$$Q_J(s_0) \equiv \frac{6}{\pi} \int_{s_{\text{th}}}^{s_0} ds \frac{\delta(s)}{s^2}.\tag{2.9a}$$

Q_Φ is obtained defining an effective phase that interpolates linearly (in t^{-1}) between the values of $\delta(t)$ at s_0 and ∞ : we write

$$\delta_{\text{eff}}(t) \equiv \pi + \left[\delta(s_0) - \pi \right] \frac{s_0}{t},\tag{2.9b}$$

and then set

$$Q_\Phi(s_0) \equiv \frac{6}{\pi} \int_{s_0}^{\infty} ds \frac{\delta_{\text{eff}}(s)}{s^2}.\tag{2.9c}$$

Finally, Q_G corrects for the difference between δ and δ_{eff} :

$$Q_G(s_0) \equiv \frac{6}{\pi} \int_{s_0}^{\infty} ds \frac{\delta(s) - \delta_{\text{eff}}(s)}{s^2}.\tag{2.9d}$$

³ Actually, the individual relations in (2.3) also hold at any t (even above s_0) provided inelasticity for the corresponding partial wave is negligible there.

Q_J , Q_Φ are known; Q_G has to be fitted or estimated. The decomposition (2.8) is equivalent to decomposing F as a product. We integrate explicitly δ_{eff} and then we can write

$$\begin{aligned} F(t) &= F(0)J(t)\Phi(t)G(t); \\ J(t) &= \exp \left\{ \frac{t}{\pi} \int_{s_{\text{th}}}^{s_0} ds \frac{\delta(s)}{s(s-t)} \right\}, \\ \Phi(t) &= e^{1-\delta_1^1(s_0)/\pi} \left(1 - \frac{t}{s_0} \right)^{[1-\delta_1^1(s_0)/\pi]s_0/t} \left(1 - \frac{t}{s_0} \right)^{-1}; \\ G(t) &= \exp \left\{ \frac{t}{\pi} \int_{s_0}^{\infty} ds \frac{\delta(s) - \delta_{\text{eff}}(s)}{s(s-t)} \right\}. \end{aligned} \quad (2.10)$$

What we know about $G(t)$ is that $G(0) = 1$, and that it is analytic except for the cut $s_0 \leq t < \infty$. The best way to take this into account is by making a conformal mapping of this cut plane into a disk, and expand in the conformal variable, $z(t)$:

$$z(t) = \frac{\frac{1}{2}\sqrt{s_0} - \sqrt{s_0 - t}}{\frac{1}{2}\sqrt{s_0} + \sqrt{s_0 - t}}. \quad (2.11a)$$

We then write^[11]

$$G(t) = 1 + A_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \quad (2.11b)$$

an expansion that will be convergent for all t inside the cut plane. We can implement the condition $G(0) = 1$, order by order, by writing $A_0 = -[c_1 z_0 + c_2 z_0^2 + c_3 z_0^3 + \dots]$, $z_0 \equiv z(t=0) = -1/3$; the expansion then reads,

$$G(t) = 1 + c_1(z + 1/3) + c_2(z^2 - 1/9) + c_3(z^3 + 1/27) + \dots, \quad (2.12)$$

the c_i being free parameters.

The contributions to the square radius Q_Φ , Q_G may be written explicitly in terms of $\delta(s_0)$, c_i as

$$Q_\Phi = \frac{3}{s_0} \left\{ 1 + \frac{\delta(s_0)}{\pi} \right\}, \quad Q_G = \frac{4}{3s_0} \left\{ c_1 - \frac{2}{3}c_2 + \dots \right\}. \quad (2.13)$$

For the electromagnetic form factor of the pion we take, following ref. 11, $s_0 = 1.1 \text{ GeV}^2$. For F_π we can fit experimental data and thus find the c_i . These data are in fact precise enough to give two terms:^[11]

$$c_1 = 0.38 \pm 0.03, \quad c_2 = -0.19 \pm 0.03 \quad [\text{For } F_\pi]. \quad (2.14)$$

The squared charge radius is then

$$\langle r_\pi^2 \rangle = 0.435 \pm 0.003 \text{ fm}^2. \quad (2.15)$$

3. Dispersive evaluation of the square radii

3.1. The electromagnetic radius of the pion

We start with a review of the evaluation of $\langle r_\pi^2 \rangle$, which will serve as a model for the other two. Although in ref. 11 the P wave phase was *deduced* from the experimental values of F_π , we here consider it as given, for $4M_\pi^2 \leq t \leq 1 \text{ GeV}^2$. Taking its value from ref. 11 one has (in fm^2)

$$Q_J = 0.195. \quad (3.1a)$$

Moreover, $\delta(s_0) = \delta_1(s_0) = 2.70$, and hence

$$Q_\Phi = 0.217. \quad (3.1b)$$

So, if we approximated $G(t) \equiv 1$ (physically, this is approximately equivalent to neglecting inelastic channels), we would underestimate the radius, but not by much, as we get

$$Q_J + Q_\Phi = 0.412,$$

which is 6% below the full value of $\langle r_\pi^2 \rangle$ as given in (2.15).

It is not easy to guess Q_G , although it is easy to understand its sign: in (2.5) we have interpolated linearly from $s_0 = 1 \text{ GeV}^2$, where $\delta(s_0) < \pi$, to $\delta(\infty) = \pi$; that is to say, systematically *below* the value π , while we know, from (2.5), that $\delta(t)$ must approach the asymptotic value of π *from above*. Thus the phase $\delta(s)$ should rise beyond π , probably around the energy of the resonance $\rho(1450)$, to reach its asymptotic behaviour (above π) after that. So we expect

$$Q_G \sim 6 \int_{s_{\text{as.}}}^{\infty} ds \frac{1}{\log s/\hat{t}}, \quad s_{\text{as.}}^{1/2} \gtrsim 1.45 \text{ GeV}. \quad (3.2)$$

Indeed, using this formula with reasonable choices of $s_{\text{as.}}$, \hat{t} , we get a value near the experimentally measured one for Q_G ; for example, we obtain the exact result, $Q_G = 0.024$, with $\hat{t} \simeq 0.3 \text{ GeV}^2$, $s_{\text{as.}}^{1/2} \simeq 1.8 \text{ GeV}$.

3.2. The mixed $K\pi$ scalar radius

We first assume the phase $\delta_0^{(1/2)}(t)$ to be given, for $t^{1/2} \leq 1.5 \text{ GeV}$, by the resonance $K^*(1430)$, whose properties we take from the PDT.^[3] Its mass is $M_* = 1412 \pm 6 \text{ MeV}$, and its width $\Gamma_* = 294 \pm 23 \text{ MeV}$; we neglect its small inelasticity ($\sim 7\%$). We write a Breit-Wigner formula for the phase:

$$\cot \delta_0^{(1/2)}(t) = \frac{t^{1/2}}{2q} (1 - s/M_*^2) B_0, \quad q = \frac{\sqrt{[s - (M_K - M_\pi)^2][s - (M_K + M_\pi)^2]}}{2s^{1/2}}$$

and $B_0 = 2q(M_*^2)/\Gamma_* = 4.15 \pm 0.35$. We take $s_0 = 1.5^2 \text{ GeV}^2$, and then we have

$$Q_J = 0.050 \pm 0.025, \quad Q_\Phi = 0.087 \pm 0.001; \quad Q_J + Q_\Phi = 0.137 \pm 0.03. \quad (3.3a)$$

This means that Q_G is large; in fact, on comparing with the experimental value, Eq. (1.5), we find

$$Q_G = 0.175 \pm 0.03. \quad (3.3b)$$

The corresponding c_1 would also be large, $c_1 = 7.6$

The sum of Q_J and Q_Φ substantially *underestimates* the value of the mixed scalar square radius: the true phase $\delta(t)$ of the form factor would have to go on growing a lot before setting to the asymptotic regime (2.5). The size of the phase necessary to produce the large Q_G required appears excessive.

An alternate possibility is the existence of a lower energy resonance (or enhancement; we denote it by κ), *below* the $K^*(1430)$, which some analyses suggest,^[12] with $M_\kappa \sim 1 \text{ GeV}$ and $\Gamma_\kappa = 400 \pm 100 \text{ MeV}$. In this case, we approximate the low energy phase, $s \leq s_0 = 1 \text{ GeV}^2$, by writing⁴

$$\cot \delta_0^{(1/2)}(t) = \frac{t^{1/2}}{2q} (1 - s/M_\kappa^2) B_\kappa, \quad B_\kappa = 1.8 \pm 0.5$$

and find

$$Q_J = 0.070 \pm 0.030, \quad Q_\Phi = 0.180 \pm 0.006; \quad Q_J + Q_\Phi = 0.250 \pm 0.030, \quad (3.4a)$$

which reproduces well the experimental number with a small Q_G , compatible with zero:

$$Q_G \sim 0.06 \pm 0.07. \quad (3.4c)$$

⁴ The Breit-Wigner parametrization (3.4) for the κ should be considered only as an effective one; it is in fact not clear that the phase would reach 90° at M_κ .

3.3. The scalar radius of the pion: bounding its value

Next we consider the quadratic scalar radius of the pion. We will, for the S0 phase below $t^{1/2} = 0.96$ GeV, take the two fits to experimental data in ref. 13: one possibility is

$$\cot \delta_0^{(0)}(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - \frac{1}{2}M_\pi^2} \frac{M_\sigma^2 - s}{M_\sigma^2} \left\{ B_0 + B_1 \frac{\sqrt{s} - \sqrt{4M_K^2 - s}}{\sqrt{s} + \sqrt{4M_K^2 - s}} \right\}, \quad k = \sqrt{s/4 - M_\pi^2};$$

$$B_0 = 21.04, \quad B_1 = 6.62, \quad M_\sigma = 782 \pm 24 \text{ MeV};$$

$$\chi^2/\text{d.o.f.} = 15.7/(19 - 3); \quad a_0^{(0)} = (0.230 \pm 0.010) M_\pi^{-1}.$$
(3.5a)

Uncorrelated errors are obtained if replacing the B_i by the parameters x, y with

$$B_0 = y - x; \quad B_1 = 6.62 - 2.59x; \quad y = 21.04 \pm 0.75, \quad x = 0 \pm 2.4.$$
(3.5b)

This will be referred to as 2Bs. Alternatively, we may take

$$\cot \delta_0^{(0)}(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - \frac{1}{2}M_\pi^2} \frac{M_\sigma^2 - s}{M_\sigma^2} \left\{ B_0 + B_1 \frac{\sqrt{s} - \sqrt{4M_K^2 - s}}{\sqrt{s} + \sqrt{4M_K^2 - s}} + B_2 \left[\frac{\sqrt{s} - \sqrt{4M_K^2 - s}}{\sqrt{s} + \sqrt{4M_K^2 - s}} \right]^2 \right\};$$

$$\chi^2/\text{d.o.f.} = 11.1/(19 - 4); \quad a_0^{(0)} = (0.226 \pm 0.015) M_\pi^{-1}$$

$$M_\sigma = 806 \pm 21, \quad B_0 = 21.91 \pm 0.62, \quad B_1 = 20.29 \pm 1.55, \quad B_2 = 22.53 \pm 3.48;$$
(3.6)

this we denote by 3Bs.

Although we think 2Bs to be more close to reality than 3Bs, and although both give very similar results, we include 3Bs because it comprises, within its errors, the S0 phase shift by Colangelo, Gasser and Leutwyler,^[6] which these authors present as very precise and incorporating results from chiral dynamics (in addition to analyticity and unitarity). We find, with self-explanatory notation,

$$Q_J(t^{1/2} \leq 0.96 \text{ GeV}; 2\text{Bs}) = 0.452 \pm 0.05 \text{ fm}^2,$$
(3.7a)

$$Q_J(t^{1/2} \leq 0.96 \text{ GeV}; 3\text{Bs}) = 0.440 \pm 0.05 \text{ fm}^2.$$
(3.7b)

Between $t^{1/2} = 0.96$ GeV and $\bar{K}K$ threshold, that we take at $t^{1/2} = 0.992$ GeV, we use a fit to experimental data as given in Eq. (3.8) of ref. 13(a), and get, in fm^2 ,

$$Q_J(0.96 \text{ GeV} \leq t^{1/2} \leq 0.992 \text{ GeV}) = 0.013 \pm 0.002.$$

We then find the numbers

$$Q_J = \begin{cases} 0.465 \pm 0.05 & [2\text{Bs}] \\ 0.453 \pm 0.05 & [3\text{Bs}]. \end{cases}$$
(3.8)

To calculate Q_Φ we take the value

$$\delta_0^{(0)}(4M_K^2) = 3.14 \pm 0.52,$$

which covers all the experimental determinations,^[14] and get $Q_\Phi = 0.237 \pm 0.02$. Therefore, we have obtained the result

$$Q_J + Q_\Phi = 0.70 \pm 0.06,$$
(3.9)

and this comprises both cases 2Bs and 3Bs.

Eq. (3.9) should be interpreted as providing a *lower bound* on $\langle r_{S,\pi}^2 \rangle$; it assumes that the phase of $F_S(s)$ does not increase for s beyond $\bar{K}K$ threshold, while, as one would deduce from the similar calculation of $\langle r_\pi^2 \rangle$, and as we will see also in the present case, $\delta(s)$ should *increase* somewhat before decreasing to its asymptotic value, $\delta(\infty) = \pi$. We have therefore found the result,

$$\langle r_{S,\pi}^2 \rangle \geq 0.70 \pm 0.06 \text{ fm}^2.$$
(3.10)

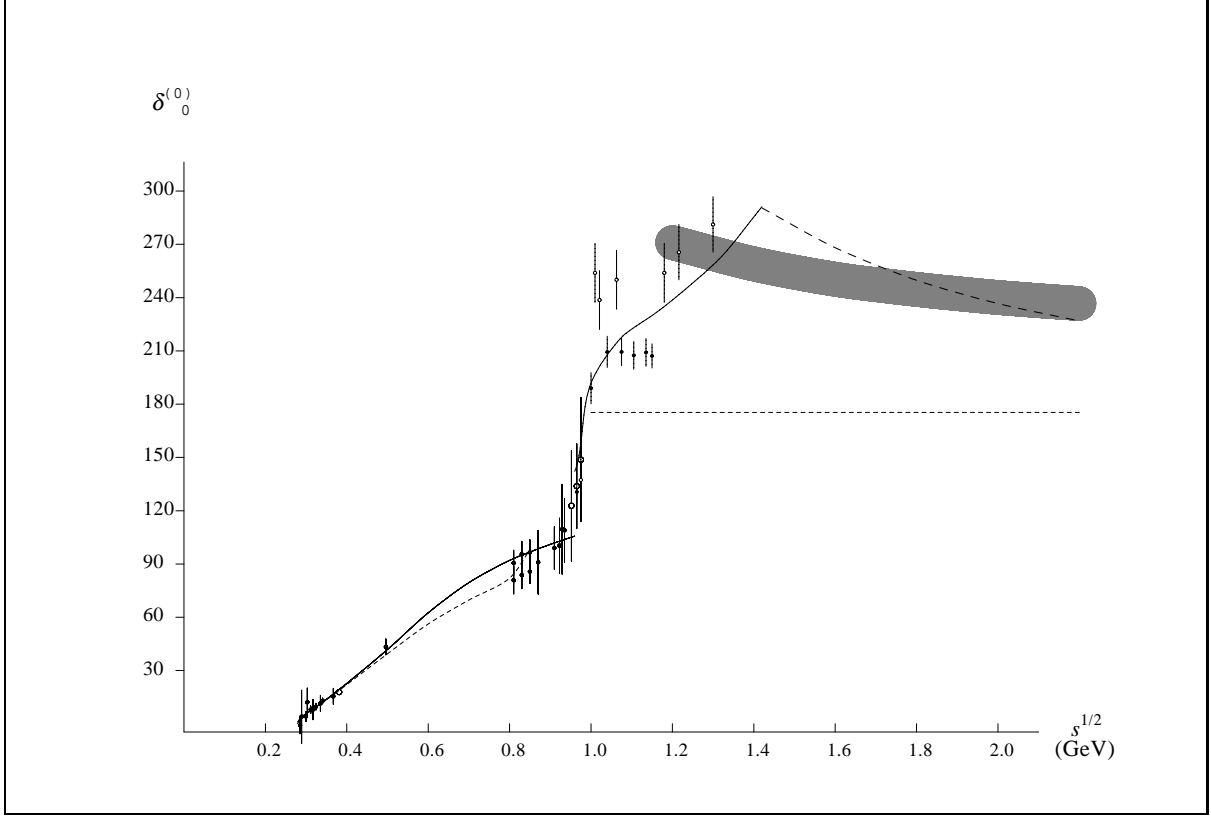


FIGURE 1. The $I = 0$, S -wave phase shift corresponding to 2Bs and Eq. (3.12) (continuous line) and the phase of ref. 6 (dotted line at low energy). Also shown experimental data points included in the 2Bs fit: from the K_{l4} experiments, and a point from $K_{2\pi}$ decay; high energy data of Protopopescu et al. (black dots), and Hyams et al., Grayer et al. (open circles), and the s -channel solution of Estabrooks and Martin. The dotted line at high energy is the line $\delta = \pi$; the dashed line is the interpolation corresponding to δ_{eff} , Eq. (2.9b). The asymptotic phase $\delta_{as.}$, with $0.1 \text{ GeV}^2 \leq \hat{t} \leq 0.35 \text{ GeV}^2$, is represented by the thick gray line.

3.4. The scalar radius of the pion: calculations

We can get a first *estimate* of the remaining quantity needed to calculate $\langle r_{S,\pi}^2 \rangle$, Q_G , by invoking SU(3) invariance. If the κ is the SU(3) partner of the σ , we indeed expect $M_\kappa \simeq 1 \text{ GeV}$. Identifying $Q_G(K\pi) \simeq Q_G(\pi)$, and using (3.4), we find an approximate number,

$$\langle r_{S,\pi}^2 \rangle \simeq 0.72 \pm 0.09 \text{ fm}^2. \quad (3.11)$$

A more sophisticated method to get Q_G is as follows. As implied by the experimental data on $\pi\pi$ scattering,^[14(b)] the inelasticity is compatible with zero (indeed, the central value is almost equal to zero) for the S_0 wave, within experimental errors, in the energy region $1.1 \text{ GeV} \leq s^{1/2} \leq 1.5 \text{ GeV}$. It thus follows that the phase of $F_S(s)$ must be approximately equal to $\delta_0^{(0)}(s)$ for $1.1 \text{ GeV} \leq t^{1/2} \leq 1.42 \text{ GeV}$.

The phases $\delta_0^{(0)}(s)$, $\delta(s)$ will likely not be equal between 0.992 GeV and 1.1 GeV; however, because this is a very short range, and the phases are equal at both endpoints (in the approximation of neglecting inelasticity there), it follows that any reasonable interpolation, e.g., a linear interpolation, will give results not very different from what one gets by taking, simply,

$$\delta(s) = \delta_0^{(0)}(s), \text{ in the full region, } 0.992 \text{ GeV} \leq s^{1/2} \leq 1.42 \text{ GeV}.$$

The distortion caused by the inelasticity being nonzero just around 1 GeV is negligible, numerically; later we will add the estimated error due to above relation being only approximately true.

We take for $\delta_0^{(0)}(s)$ the fit to experimental data in ref. 13(a), Eq. (3.8),

$$\cot \delta_0^{(0)}(s) = c_0 \frac{(s - M_\sigma^2)(M_f^2 - s)|k_2|}{M_f^2 s^{1/2} k_2^2}; \quad k_2 = \sqrt{s/4 - M_K^2}; \quad (3.12)$$

$$s^{1/2} \geq 0.96 \text{ GeV}; \quad c_0 = 1.36 \pm 0.05, \quad M_\sigma = 0.802 \text{ GeV}, \quad M_f = 1.32 \text{ GeV}.$$

The corresponding $\delta_0^{(0)}$ is shown in Fig. 1. We write, choosing the 2Bs fit for the S0 wave below $\bar{K}K$ threshold,

$$Q_J(s_0 = 1.42^2 \text{ GeV}^2) = Q_J(0.992^2 \text{ GeV}^2) + Q_J(0.992^2 \text{ to } 1.42^2 \text{ GeV}^2),$$

and we have, in fm^2 ,

$$Q_J(0.992^2 \text{ GeV}^2) = 0.465 \pm 0.05, \quad Q_J(0.992^2 \text{ to } 1.42^2 \text{ GeV}^2) = 0.162 \pm 0.002; \\ Q_J(s_0 = 1.42^2 \text{ GeV}^2) = 0.627 \pm 0.05.$$

We note that the error in $Q_J(0.992^2 \text{ to } 1.42^2 \text{ GeV}^2)$ is only the error coming from c_0 in (3.12); the error due to neglect of the inelasticity we expect to be much larger, of the order of 10% to 15%.

In our present approximation, neglecting inelasticity below 1.42 GeV, we have $\delta(1.42^2 \text{ GeV}^2) = \delta_0^{(0)}(1.42^2 \text{ GeV}^2) = 5.10 \pm 0.03$, hence

$$Q_\Phi(s_0 = 1.42^2 \text{ GeV}^2) = 0.152 \pm 0.001.$$

Thus, in this calculation, and adding the estimated error due to neglect of inelasticity between 1 GeV and 1.42 GeV, we find

$$\langle r_{S,\pi}^2 \rangle = 0.78 \pm 0.06 \text{ (St.)}_{-0.07}^{+0} \text{ (Inelast.) fm}^2, \quad (3.13)$$

compatible with (3.11). Although the central value here is probably displaced upwards (after all, there is *some* inelasticity), so that (3.13) should probably be considered more like an *upper* bound, we emphasize that this value is *attainable*. Because experimental data are, at rather less than 1σ , compatible with zero inelasticity, it follows that any realistic estimate for $\langle r_{S,\pi}^2 \rangle$ must have error bars containing the value (3.13). This is one of the reasons why a two-channel evaluation is superfluous.

It is suggestive that, if we take the asymptotic formula (2.5) for $\delta(t)$, with \hat{t} between 0.1 GeV^2 and 0.35 GeV^2 , then this coincides, on the average and to a 10% accuracy, with the $\delta_0^{(0)}(t)$, $\delta_{\text{eff}}(t)$ [the second as given by (3.12), (2.9b) with $s_0 = 1.42^2 \text{ GeV}^2$], for $t^{1/2}$ between 1.1 GeV and 2 GeV; see Fig. 1. This lends additional credence to our calculation (3.13), and it also suggests a different method of evaluation. We note that the asymptotic expression,

$$\delta_{\text{as.}}(s) = \pi \left\{ 1 + \frac{\nu}{\log s/\hat{t}} \right\},$$

with $\nu = 1$, intersects the phase given by Eq. (3.12) at $s \simeq 1.35^2 \text{ GeV}^2$, for $0.1 \text{ GeV}^2 \leq \hat{t} \leq 0.35 \text{ GeV}^2$. We can then use (3.12) for $s \leq 1.35^2 \text{ GeV}^2$ and the asymptotic expression $\delta_{\text{as.}}$, with $0.1 \text{ GeV}^2 \leq \hat{t} \leq 0.35 \text{ GeV}^2$, for $s \geq 1.35^2 \text{ GeV}^2$. This gives

$$\langle r_{S,\pi}^2 \rangle = 0.76 \pm 0.06 \text{ fm}^2, \quad (3.14)$$

i.e., a result almost identical to (3.13). Indeed, any reasonable interpolation between the asymptotic phase and the low energy one would give a similar result.

Taking into account this, as well as the previous results, we get what we consider a reliable value for the scalar radius by writing

$$\langle r_{S,\pi}^2 \rangle = 0.75 \pm 0.06 \text{ fm}^2, \quad (3.15)$$

which encompasses (3.10), (3.11), (3.13) and (3.14).

4. Discussion

We first say a few words about $\langle r_{S,K\pi}^2 \rangle$. The central experimental value, $\langle r_{S,K\pi}^2 \rangle_{\text{exp.}} = 0.31 \pm 0.06 \text{ fm}^2$, is 2σ above the theoretical prediction of Gasser and Leutwyler,^[2] $0.20 \pm 0.05 \text{ fm}^2$. It would seem that the errors were underestimated by a factor ~ 2 by these authors. The experimental number, together with our dispersive evaluation, suggest the existence of the κ enhancement.^[12]

We next turn to the scalar radius. The experiment-based evaluation of Donoghue, Gasser and Leutwyler,^[5,6] $\langle r_{S,\pi}^2 \rangle_{\text{DGL}} = 0.61 \pm 0.04 \text{ fm}^2$, lies *below* our lower bound, (3.10), and well below our best estimate, $0.75 \pm 0.07 \text{ fm}^2$. To get a value as low as that of these authors, one would have to assume that, contrary to indications from $\pi\pi$ scattering, the phase of $F_S(t)$ would *decrease* above $t^{1/2} = 1 \text{ GeV}$ to about $1/2$ of its asymptotic value; and that this continues to hold up to a very high energy.⁵ This is of course a very unlikely behavior and, what is worse for a calculation based upon experimental phase shifts, it is incompatible with what one may get within experimental errors, as proved by our calculation (3.13). From this it follows that the chiral dynamics calculation at one loop,^[2] $\langle r_{S,\pi}^2 \rangle = 0.55 \pm 0.15 \text{ fm}^2$, lies clearly below the value suggested by experiment, Eq. (3.15).

To finish we say a few words on the value of the chiral constant \bar{l}_4 , and the connection with $\pi\pi$ parameters. We present a few values for \bar{l}_4 , all of them, however, using the DGL^[5] value for $\langle r_{S,\pi}^2 \rangle$:

$$\begin{aligned} \bar{l}_4 &= 4.4 \pm 0.3 \quad [\text{BCT}]; & \bar{l}_4 &= 4.2 \pm 0.2 \quad [\text{ABT}]; \\ \bar{l}_4 &= 4.2 \pm 1.0 \quad [\text{DFGS}]; & \bar{l}_4 &= 4.4 \pm 0.2 \quad [\text{Best CGL value; with } \pi\pi \text{ info}]. \end{aligned} \quad (4.1)$$

Here CGL is ref. 6, ABT and BCT are in ref. 15, and DFGS denotes the paper by Descotes et al.^[16] This is to be compared to what one gets, at one loop accuracy, from our results here:

$$\begin{aligned} \bar{l}_4 &= 5.4 \pm 0.5, & \text{with our best estimate, (3.15);} \\ \bar{l}_4 &\geq 5.1 \pm 0.4, & \text{with our bound (3.10).} \end{aligned} \quad (4.2)$$

Two loop corrections to the various form factors have been evaluated in ref. 17; see also ref. 6. If we accepted the value given by Colangelo, Gasser and Leutwyler^[6] for the higher order correction, $\delta\bar{l}_4 \simeq -0.25$, we would still find that their estimate in (4.1) is too low. This should have implications for the accuracy of their description of $\pi\pi$ scattering, as already mentioned by Descotes et al.,^[16] who get, from fits including realistic errors for the $\pi\pi$ phase shifts, much larger errors for \bar{l}_4 than the rest.

In what respects to the connection with low energy $\pi\pi$ scattering parameters, our value here for $\langle r_{S,\pi}^2 \rangle$ is in reasonable agreement with the D wave scattering lengths deduced in ref. 13, using the Froissart–Gribov representation with correct Regge expressions at high energy. From the relation

$$1 + \frac{1}{3}M_\pi^2 \langle r_{S,\pi}^2 \rangle = 24\pi f_\pi^2 \left\{ a_1 - \frac{10}{3}M_\pi^2 \left(a_2^{(0)} - \frac{5}{2}a_2^{(2)} \right) \right\} M_\pi - \frac{19}{576} \frac{M_\pi^2}{\pi^2 f_\pi^2},$$

and using the numbers⁶ of ref. 13 for a_1 and the $a_2^{(I)}$, we find

$$\langle r_{S,\pi}^2 \rangle = 0.83 \pm 0.17 \text{ fm}^2,$$

⁵ It is difficult to point out where lies the failure in the calculation of Donoghue, Gasser and Leutwyler, as it is of the “black-box” type. However, a hint is obtained from their statement (p. 356 of ref. 5) that the values of the phases above $s^{1/2} = 1.4 \text{ GeV}$ do not significantly affect their results. Contrary to this, our explicit calculations show that the contributions to (2.7) from energies above 1.4 GeV are *large*: of 20% for $\langle r_{S,\pi}^2 \rangle$, and of 27% for the electromagnetic radius, $\langle r_\pi^2 \rangle$, where one can check the estimate against experiment. They also provide 28% of $\langle r_{S,K\pi}^2 \rangle$.

⁶ Specifically, we take $a_2^{(0)} = (18.0 \pm 0.2) \times 10^{-4} M_\pi^{-5}$, $a_2^{(2)} = (2.2 \pm 0.2) \times 10^{-4} M_\pi^{-5}$ and we have improved the value of a_1 by combining the independent determinations of this quantity from F_π and with the Froissart–Gribov representation to get $a_1 = (38.3 \pm 0.8) \times 10^{-3} M_\pi^{-3}$.

compatible with our best value (3.14). If we had used the relation between $\langle r_{S,\pi}^2 \rangle$ and the S wave phase shifts,

$$1 + \frac{1}{3} M_\pi^2 \langle r_{S,\pi}^2 \rangle = \frac{4\pi}{3} \frac{f_\pi^2}{M_\pi^2} \left\{ 2a_0^{(0)} - 5a_0^{(2)} \right\} - \frac{41}{192} \frac{M_\pi^2}{\pi^2 f_\pi^2},$$

we would have obtained somewhat larger numbers,

$$\langle r_{S,\pi}^2 \rangle = 1.26 \pm 0.26 \text{ fm}^2, \quad 1.14 \pm 0.21 \text{ fm}^2$$

depending on whether we use the scattering lengths themselves or the value for the combination $2a_0^{(0)} - 5a_0^{(2)}$ from the dispersive integral in the Olsson sum rule [ref. 13(a), Eq. (4.3)]. These values are a bit more than 1.5σ above (3.14): doubtlessly because the higher order corrections are larger for the relation involving the S waves scattering lengths.

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References

- 1 Gasser, J., and Leutwyler, H., *Ann. Phys. (N.Y.)*, **158**, 142 (1984). Second order calculations of both the pion and mixed form factors have appeared recently: Moussallam, B., *Eur. Phys. J.*, **C14**, 111 (2000), Ananathanarayan, B., Buttiker, P., and Moussallam, B., *Eur. Phys. J.*, **C22**, 133 (2000), Jamin, M., Oller, J. A., and Pich, A., *Nucl. Phys.*, **B266**, 279 (2002), and Bijmens, J., and Dhonte, P., LU TP 03-32 (2003), hep-ph/0307044. They do not add anything essential in connection with what interests us here.
- 2 Gasser, J., and Leutwyler, H., *Nucl. Phys.*, **B250**, 517 (1985).
- 3 Particle Data Tables: D. E. Groom et al., *Eur. Phys. J.*, **C15**, 1 (2000).
- 4 Artemov, V. M., et al., *Phys. Atom. Nucl.*, **60**, 2023 (1997); Whitman, R., et al., *Phys. Rev.*, **D21**, 652 (1980); Heintze, J., et al., *Phys. Letters*, **70B**, 482 (1977); Braun, et al., *Nucl. Phys.*, **B89**, 210 (1975).
- 5 Donoghue, J. F., Gasser, J., and Leutwyler, H., *Nucl. Phys.*, **B343**, 341 (1990).
- 6 Colangelo, G., Gasser, J., and Leutwyler, H., *Nucl. Phys.*, **B603**, 125 (2001).
- 7 Atkinson, D., Mahoux, G., and Ynduráin, F. J., *Nucl. Phys.*, **B54**, 263 (1973); **B98**, 521 (1975). This ambiguity is far from being academic. For example, experimental phase shift analyses give a P wave in $\pi\pi$ scattering both essentially elastic, and clearly smaller than π for $s \lesssim 4\text{ GeV}^2$. This two properties, however, are unlikely to hold simultaneously: this energy should be already asymptotic, and then the theorem (2.5) implies that one should have $\delta_1(t) = \delta(t) > \pi$.
- 8 Ynduráin, F. J., *Nucl. Phys.*, **B88**, 316 (1975); Aguilar-Benítez, M., et al., *Nucl. Phys.*, **B140**, 73 (1978).
- 9 Brodsky, S. J., and Farrar, G., *Phys. Rev. Lett.*, **31**, 1153 (1973); Farrar, G., and Jackson, D. R., *Phys. Rev. Lett.*, **43**, 246 (1979). See also Brodsky, S. J., and Lepage, G. P., *Phys. Rev.* **D22**, 2157 (1980); Duncan, A., in *Topical Questions in QCD*, *Phys. Scripta*, **23**, No. 5 (1981).
- 10 Muskhelishvili, N. I., *Singular Integral Equations*, Nordhoff, Groningen (1958). A detailed discussion tailored for the present cases, may be found in Ynduráin, F. J., “Low energy pion interactions”, FTUAM 02-28 (hep-ph/0212282).
- 11 de Trocóniz, J. F., and Ynduráin, F. J., *Phys. Rev.*, **D65**, 093001 (2002).
- 12 Theory: Oller, J. A., Oset, E., and Peláez, J. R., *Phys. Rev.*, **D59**, 074001 (1999) and Gómez-Nicola, A. and Peláez, J. R., *Phys. Rev.*, **D65**, 054009 (2002). Experiment: Aitala, E. M., et al., *Phys. Rev. Lett.*, **89**, 12801 (2002). We take the width and mass of the κ from this last reference. Although this resonance has not made it to the PDT, it appears very likely that (bona fides resonance or not) the $S_{\frac{1}{2}}$ phase for $K\pi$ scattering is strongly enhanced at an energy a little below 1 GeV.
- 13 (a) Peláez, J. R., and Ynduráin, F. J., hep-ph/0304067 (to be published in *Phys. Rev. D*). See also (b) Peláez, J. R., and Ynduráin, F. J., FTUAM 03-14, 2003 (especially Appendix A for phase shifts).
- 14 (a) Protopopescu, S. D., et al., *Phys. Rev.* **D7**, 1279 (1973); (b) Hyams, B., et al., *Nucl. Phys.* **B64**, 134 (1973); Grayer, G., et al., *Nucl. Phys.* **B75**, 189 (1974). See also the analysis of the same experimental data in Estabrooks, P., and Martin, A. D., *Nucl. Physics*, **B79**, 301 (1974).
- 15 (BCT) Bijmens, J., Colangelo, G., and Talavera, G., *JHEP*, **9805**, 014 (1998); (ABT) Amorós, G., Bijmens, J., and Talavera, G., *Nucl. Phys.*, **B585**, 293 (2000).
- 16 Descotes, S., Fuchs, N. H., Girlanda, L., and Stern, J., *Eur. Phys. J. C*, **15**, 469 (2002).
- 17 Gasser, J., and Meißner, U. G., *Nucl. Phys.*, **B357**, 90 (1991); Colangelo, G., Finkelmeir, M., and Urech, R., *Phys. Rev.*, **D54**, 4403 (1996); Frink, M., Kubis, B., and Meißner, U. G., *Eur. Phys. J.*, **C25**, 259 (2002).